BENJAMINI–SCHRAMM CONTINUITY OF ROOT MOMENTS OF GRAPH POLYNOMIALS

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Abstract. Recently, M. Abért and T. Hubai studied the following problem. The chromatic measure of a finite simple graph is defined to be the uniform distribution on its chromatic roots. Abért and Hubai proved that for a Benjamini-Schramm convergent sequence of finite graphs, the chromatic measures converge in holomorphic moments. They also showed that the normalized log of the chromatic polynomial converges to a harmonic real function outside a bounded disc.

In this paper we generalize their work to a wide class of graph polynomials, namely, multiplicative graph polynomials of bounded exponential type. A special case of our results is that for any fixed complex number v_0 the measures arising from the Tutte polynomial $Z_{G_n}(z, v_0)$ converge in holomorphic moments if the sequence (G_n) of finite graphs is Benjamini–Schramm convergent. This answers a question of Abért and Hubai in the affirmative. Even in the original case of the chromatic polynomial, our proof is considerably simpler.

1. Introduction — background and main results

This paper generalizes certain results of Sokal [14], Borgs, Chayes, Kahn and Lovász [3] and, most importantly, Abért and Hubai [1]. In this section we briefly recall these results and state our main theorems: 1.6 and 1.10.

Let $G = (V, E)$ be a finite, simple, undirected graph. (Note that we follow the usual notations. However, if some notation is unclear, the reader may find its meaning at the end of this section.) A map $f : V \to \{1, 2, \dots, q\}$ is a *proper coloring* if for all edges $(x, y) \in E$ we have $f(x) \neq f(y)$. For a positive integer q let $ch(G, q)$ denote the number of proper colorings of G with q colors. Then it turns out that $ch(G, q)$ is a polynomial in q. It is called the chromatic polynomial [11]. Let us call the roots of the chromatic polynomial *chromatic roots*. The *chromatic measure* is the probability measure μ_G on $\mathbb{C}$ given by the uniform distribution on the chromatic roots.

1.1. **The Sokal bound.**

More than a decade ago, Alan Sokal proved the following theorem.

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Theorem 1.1 (Sokal [14]). *Let G be a graph of largest degree Δ. Then the absolute value of any root of the chromatic polynomial of G is at most $C\Delta$, where the constant C is less than 8.*

Later Jackson, Procacci and Sokal [9] extended this result to a more general graph polynomial, the Tutte polynomial. In Theorem 1.6, we shall further generalize Theorem 1.1 to a wide class of graph polynomials. This will set the stage for our main result, Theorem 1.10.

Definition 1.2. A *graph polynomial* is a mapping f that assigns to every finite simple graph G a polynomial $f(G,x) \in \mathbb{C}[x]$. The graph polynomial f is *monic* if the polynomial $f(G,x)$ is monic of degree $|V(G)|$ for all graphs G.

Definition 1.3. We say that the graph polynomial f is *of exponential type* if $f(\emptyset, x) = 1$ and for every graph $G = (V(G), E(G))$ we have

$$\sum_{S \subseteq V(G)} f(S,x) f(G-S,y) = f(G, x+y), \tag{1.1}$$

where $f(S,x) = f(G[S],x)$ and $f(G-S,y) = f(G[V(G) \setminus S], y)$ are the polynomials of the subgraphs of G induced by the sets S and $V(G) \setminus S$, respectively.

In the next definition, we introduce a boundedness condition that is relatively easy to check and implies a Sokal-type bound on the roots.

Definition 1.4. Assume that

$$f(G,x) = \sum_{k=0}^{n} a_k(G) x^k$$

is a monic graph polynomial of exponential type. Assume that there is a function $R : \mathbb{N} \to [0,\infty)$ not depending on G such that for any graph G with all degrees at most Δ, any vertex $v \in V(G)$ and any $t \geq 1$ we have

$$\sum_{\substack{v \in S \subseteq V(G) \\ |S| = t}} |a_1(G[S])| \leq R(\Delta)^{t-1}.$$

In this case we say that the graph polynomial $f(G,x)$ is *of bounded exponential type* with bounding function R.

Remark 1.5. Examples of graph polynomials of bounded exponential type include the chromatic polynomial, the Tutte polynomial, the (modified) matching polynomial, the adjoint polynomial and the Laplacian characteristic polynomial. Cf. Remark 5.4 and Proposition 2.1.

By imitating Sokal's proof, we will prove the following extension of his theorem (note, however, that our constant is much weaker).

Theorem 1.6. *Let $f(G,x)$ be a graph polynomial of bounded exponential type with bounding function $R > 0$. Let G be a graph of largest degree $\leq \Delta$. Then the absolute value of any root of $f(G,z)$ is less than $cR(\Delta)$, where $c < 7.04$.*

Definition 1.7. We say that the graph polynomial f *has bounded roots* if there exists a function $\tilde{R} : \mathbb{N} \to (0, \infty)$ such that for every $\Delta \in \mathbb{N}$ and every graph G with all degrees at most Δ, the roots of the polynomial $f(G, z)$ have absolute value less than $\tilde{R}(\Delta)$.

According to Sokal's Theorem 1.1, the chromatic polynomial has bounded roots. By Theorem 1.6, every graph polynomial of bounded exponential type has bounded roots.

1.2. **Convergence.** Recently, M. Abért and T. Hubai [1] studied the behaviour of chromatic measures in Benjamini–Schramm convergent graph sequences.

We recall the definition of Benjamini–Schramm convergence. For a finite graph G, a finite rooted graph α and a positive integer r, let $\mathbb{P}(G, \alpha, r)$ be the probability that the r-ball centered at a uniform random vertex of G is isomorphic to α. We say that a graph sequence (G_n) of bounded degree is Benjamini-Schramm convergent if for all finite rooted graphs α and $r > 0$, the probabilities $\mathbb{P}(G_n, \alpha, r)$ converge. This means that one cannot distinguish G_n and $G_{n'}$ for large n and n' by sampling them from a random vertex with a fixed radius of sight.

Our main interest is in the behaviour of the roots of $f(G_n, z)$ as $n \to \infty$. For our main result, we need to put two further restrictions on f.

Definition 1.8. The graph polynomial f is *isomorphism-invariant* if $G_1 \simeq G_2$ implies $f(G_1, x) = f(G_2, x)$.

Definition 1.9. The graph polynomial f is *multiplicative* if

$$f(G_1 \uplus G_2, x) = f(G_1, x) f(G_2, x),$$

where $G_1 \uplus G_2$ denotes the disjoint union of the graphs G_1 and G_2.

The examples in Remark 1.5 are all isomorhism-invariant and multiplicative (in addition to being monic, of exponential type, and having bounded roots).

Now we are ready to state our main result.

Theorem 1.10. *Let f be an isomorphism-invariant monic multiplicative graph polynomial of exponential type. Assume that f has bounded roots.*

Let (G_n) be a Benjamini-Schramm convergent graph sequence. Let $K \subset \mathbb{C}$ be a compact set containing all roots of $f(G_n, x)$ for all n, such that $\mathbb{C} \setminus K$ is connected.

(a) *For a graph G, let μ_G be the uniform distribution on the roots of $f(G, x)$. Then for every continuous function $g : K \to \mathbb{R}$ that is harmonic on the interior of K, the sequence*

$$\int_K g(z) d\mu_{G_n}(z)$$

converges.

Moreover, for any open set $\Omega \subseteq \mathbb{R}^d$ and any continuous function $g : K \times \Omega \to \mathbb{R}$ that is harmonic on the interior of K for any fixed

$\xi \in \Omega$ and harmonic on Ω for any fixed $z \in K$, the sequence

$$\int_K g(z,\xi) d\mu_{G_n}(z)$$

converges, locally uniformly in $\xi \in \Omega$, to a harmonic function on Ω.

(b) *For $\xi \in \mathbb{C} \setminus K$, let*

$$t_n(\xi) = \frac{\log |f(G_n, \xi)|}{|V(G_n)|}.$$

Then $t_n(\xi)$ converges to a harmonic function locally uniformly on $\mathbb{C} \setminus K$.

The integral in (a) is called a *harmonic moment* of the roots of $f(G_n, z)$. Its convergence is referred to as the 'Benjamini–Schramm continuity' or 'testability' of the moment. The fraction in (b) is called the *entropy per vertex* or the *free energy* at ξ.

The phenomenon described in the Theorem was discovered, in the case of the chromatic polynomial, by M. Abért and T. Hubai. The main achievement of their paper [1] was to state and prove Theorem 1.10 for the chromatic polynomial (in a slightly different form: they used a disc in place of our set K, they used holomorphic functions in place of our harmonic ones, they took $\log f$ in place of $\log |f|$ in (b) — all these are inessential differences). They thus answered a question of Borgs and generalized a result of Borgs, Chayes, Kahn and Lovász [3], who had proved convergence of $(\log ch(G_n, q))/|V(G_n)|$ at large positive integers q.

One may naively hope for the weak convergence of the measures μ_{G_n} arising from the roots. As remarked by Abért and Hubai, this is easily seen to be false already for the chromatic polynomial. Indeed, paths and circuits together form a Benjamini–Schramm convergent sequence, but the chromatic measures of paths tend to the Dirac measure at 1 and the chromatic measures of circuits tend to the uniform measure on the unit circle centered at 1. So it was a crucial observation that the useful relaxation is to consider the convergence of the holomorphic moments.

To prove the convergence of the holomorphic moments, Abért and Hubai [1] showed (essentially) that for a finite graph G and for every k, the number

$$p_k(G) = |V(G)| \int_K z^k d\mu_G(z)$$

can be expressed as a fixed linear combination of the numbers $H(G)$, where the H are non-empty **connected** finite graphs and $H(G)$ denotes the number of subgraphs of G isomorphic to H.

To show this, they determined an exact expression for the power sum $p_k(G)$ of the roots in terms of homomorphism numbers. Our approach is a bit different: without determining the exact expression, we use multiplicativity of the graph polynomial to show that the coefficient of $H(G)$ for non-connected H must be 0. (In fact, we can determine the exact expression too with a little extra work: see Remark 6.7.) The argument is simplified by using subgraph counting, which is equivalent to injective homomorphism numbers, instead of ordinary homomorphism numbers.

Besides seeking a proof without lengthy calculations, we felt it desirable to clarify which properties of the chromatic polynomial are needed to prove such a result. We wished to grasp the right concepts for the generalization of the Abért–Hubai result. This is achieved by the definitions we have given in this section.

This paper is organized as follows. In Section 2 we recall the Tutte polynomial and explain how and why Theorem 1.10 applies to it. In Section 3 we recall some basic facts related to subgraph counting. In Section 4 we prove Theorem 4.6, which will clearly cover Theorem 1.10 for the most interesting graph polynomials, including the chromatic polynomial and the Tutte polynomial. The fact that Theorem 4.6 covers Theorem 1.10 in general will be shown in Section 5.

In Section 5 we study graph polynomials of exponential type. This part will consist of two subsections. In the first subsection we characterize graph polynomials of exponential type, describe some of their fundamental properties and give further examples. In the second subsection we prove Theorem 1.6.

In Section 6 we introduce the notion of 2-multiplicativity, which is a stronger version of multiplicativity. This section is not needed for the main results, but it makes the picture more complete.

Finally, we end the paper by some concluding remarks.

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We end this section by setting down the notations.

Notations. Throughout the paper we will consider only finite simple graphs. Connected graphs are assumed to have at least one vertex. 2-connected graphs are assumed to have at least two vertices. Let $\mathcal{G}$, $\mathcal{C}$ and $\mathcal{C}_2$ denote the class of graphs, connected graphs and 2–connected graphs, respectively. For a graph G let $k(G)$ denote the number of connected components of G.

We will follow the usual notations: G is a graph, $V(G)$ is the set of its vertices, $E(G)$ is the set of its edges, $e(G)$ denotes the number of edges, $N(x)$ is the set of the neighbors of x, $|N(v_i)| = \deg(v_i) = d_i$ denotes the degree of the vertex v_i. We will also use the notation $N[v]$ for the closed neighborhood $N(v) \cup \{v\}$.

For $S \subseteq V(G)$ the graph $G - S$ denotes the subgraph of G induced by the vertices $V(G) \setminus S$ while $G[S]$ denotes the subgraph of G induced by the vertex set S. In case of a graph polynomial f we write $f(S, x)$ instead of $f(G[S], x)$ if the graph G is clear from the context. If G is a graph with vertex set V and edge set E then $G(V, E')$ denotes the spanning subgraph with vertex set V and edge set $E' \subseteq E$.

If S is a set then $|S|$ denotes its cardinality. The notation $S_1 \uplus S_2 = V$ means that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = V$.

2. The Tutte polynomial

A main example that Theorem 1.10 applies to is the Tutte polynomial, answering a question raised by Abért and Hubai. The Tutte polynomial of a graph G is defined as follows:

$$T(G,x,y)=\sum_{A\subseteq E}(x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|},$$

where $k(A)$ denotes the number of connected components of the graph (V,A). In statistical physics one often studies the following form of the Tutte polynomial:

$$Z_G(q,v)=\sum_{A\subseteq E}q^{k(A)}v^{|A|}.$$

The two forms are essentially equivalent:

$$T(G,x,y)=(x-1)^{-k(E)}(y-1)^{-|V|}Z_G((x-1)(y-1),y-1).$$

Both forms have several advantages. For instance, it is easy to generalize the latter one to define the multivariate Tutte polynomial. Let us assign a variable v_e to each edge and set

$$Z_G(q,\underline{v})=\sum_{A\subseteq E}q^{k(A)}\prod_{e\in A}v_e.$$

We will work with the form $Z_G(q,v)$, because the degree of this polynomial with respect to the variable q is $|V(G)|$. So it will be a bit more convenient to work with this definition.

Note that the chromatic polynomial of the graph G is

$$ch(G,x)=Z_G(x,-1)=(-1)^{|V|-k(G)}x^{k(G)}T(G,1-x,0).$$

Proposition 2.1. (a) *The chromatic polynomial is multiplicative and of bounded exponential type with $R(\Delta)=e\Delta$.*
(b) *For any $v_0\in\mathbb{C}$, the Tutte polynomial $Z_G(q,v_0)$ is multiplicative and of bounded exponential type with $R(\Delta)=e\Delta(1+|v_0|)^{\Delta}$.*

Proof. Multiplicativity and exponential type are evident from the definition. The boundedness of these polynomials can be found in the papers [14, 9] in a stronger form. □

Corollary 2.2. *Theorem 1.10 applies to the Tutte polynomial $f(G,q)=Z_G(q,v_0)$ for any fixed $v_0\in\mathbb{C}$ and, in particular, to the chromatic polynomial.*

This recovers the main result of [1], and answers a question raised there.

3. Preliminaries on subgraph counting

We write $H(G)$, resp. $H^*(G)$ for the number of subgraphs, resp. induced subgraphs of G isomorphic to H. Thus, by a slight abuse of notation, H denotes not only a graph but also an integer-valued, isomorphism-invariant function on graphs.

It is well known that if H runs over the isomorhism classes of graphs, then the functions $H:G\mapsto H(G)$, resp. $H^*:G\mapsto H^*(G)$ are linearly independent. See, for example, Lemma 4.1 in [3].

Any family $\mathcal{H}$ of isomorphism types of graphs gives rise to the $\mathbb{C}$–vector space $\mathbb{C}\mathcal{H}$, resp. $\mathbb{C}\mathcal{H}^*$ generated by the functions H, resp. H^*, where $H \in \mathcal{H}$:

$$\mathbb{C}\mathcal{H} = \left\{ \sum_{H \in \mathcal{H}} c_H H(\cdot) \mid c_H \in \mathbb{C} \right\},$$

where all sums are finite.

Definition 3.1. The family $\mathcal{H}$ is *ascending* if it is closed under the operation of adding new edges (but no new vertices) to a graph.

We need the following well-known facts.

Fact 3.2. *If $\mathcal{H}$ is ascending, then $\mathbb{C}\mathcal{H} = \mathbb{C}\mathcal{H}^*$.*

Fact 3.3.

$$H_1(G) \cdot H_2(G) = \sum_H c^H_{H_1,H_2} H(G),$$

where the coefficient $c^H_{H_1,H_2}$ is the number of decompositions of H as a (not necessarily disjoint) union of H_1 and H_2. This number is a non-negative integer and it is zero for all but finitely many isomorphism types H.

Corollary 3.4. *If $\mathcal{H}$ is closed under unions, then $\mathbb{C}\mathcal{H}$ is a ring.*

4. Multiplicative lemma

In this section it is shown how multiplicativity implies Benjamini–Schramm convergence. The main result will be Theorem 4.6. Its proof is made simple by the ideas included in Lemmas 4.2 and 4.3.

Definition 4.1. A function p defined on graphs is *additive* if

$$p(G_1 \uplus G_2) = p(G_1) + p(G_2)$$

holds for any two graphs G_1 and G_2.

Lemma 4.2. *(Additive lemma.) A function $p \in A\mathcal{G}$ is additive if and only if $p \in A\mathcal{C}$.*

Proof. The 'if' part follows from the evident fact that $H(G)$ is additive (in G) for any connected H.

We prove the 'only if' part. Let

$$p(G) = \sum_H c_H \cdot H(G)$$

for every graph G. We know that p is additive and we need to prove that $c_H = 0$ for all non-connected graphs H. Using the 'if' part, we may assume that $c_H = 0$ for all connected graphs H.

Let H be a non-connected graph. Then H is the disjoint union of graphs H_1 and H_2 with at least 1 vertex. By induction, we may assume that $c_{H'} = 0$ for all proper subgraphs H' of H. Then

$$p(H_i) = \sum_{H'} c_{H'} \cdot H'(H_i) = \sum 0 = 0$$

for $i = 1, 2$ and

$$p(H) = \sum_{H'} c_{H'} \cdot H'(H) = c_H \cdot H(H) = c_H.$$

By additivity we get $c_H = 0$. $\square$

Lemma 4.3. *(Multiplicative lemma.) Assume that f is a multiplicative graph polynomial such that $f(G,x)$ is not the zero polynomial for any graph G. Let $\lambda_1(G), \dots, \lambda_n(G)$ denote the roots of the polynomial $f(G,x)$, where $n = \deg f(G,x)$. Assume that for some k the k–th power sum p_k of the roots is in $\mathbb{C}\mathcal{G}$, i.e., there exist constants $c_k(H)$, nonzero for only finitely many isomorphism types H, such that*

$$p_k(G) = \sum_{i=1}^{n} \lambda_i(G)^k = \sum_H c_k(H) \cdot H(G)$$

for every graph G. Then $p_k \in \mathbb{C}\mathcal{C}$, i.e., $c_k(H) = 0$ for non-connected graphs H.

Proof. The function $G \mapsto p_k(G)$ is additive, so the statements follow from Lemma 4.2. $\square$

The main result of this section, Theorem 4.6, will be an easy corollary of Lemma 4.3.

We recall the following well-known fact concerning Benjamini-Schramm convergence.

Fact 4.4. *Let (G_n) be a graph sequence of bounded degree. Then (G_n) is Benjamini-Schramm convergent if and only if for every (finite, non-empty) connected graph H, the sequence*

$$\frac{H(G_n)}{|V(G_n)|}$$

converges.

Corollary 4.5. *Let (G_n) be a graph sequence of bounded degree. Then (G_n) is Benjamini-Schramm convergent if and only if for every $p \in \mathbb{C}\mathcal{C}$, the sequence*

$$\frac{p(G_n)}{|V(G_n)|}$$

converges.

Theorem 4.6. *Let f be a multiplicative monic graph polynomial with bounded roots. We also assume that*

$$f(G,x) = \sum_{k=0}^{n} (-1)^k e_k(G) x^{n-k},$$

where $n = |V(G)|$ and all coefficients $e_k \in \mathbb{C}\mathcal{G}$. Let (G_n) be a Benjamini-Schramm convergent graph sequence. Let $K \subset \mathbb{C}$ be a compact set containing all roots of $f(G_n, x)$ for all n, such that $\mathbb{C} \setminus K$ is connected. Then the statements (a) and (b) of Theorem 1.10 hold.

Remark 4.7. Our main result, Theorem 1.10 follows from Theorem 4.6, using also the fact that $e_k \in \mathbb{C}\mathcal{G}$ if f is isomorphism-invariant and of exponential type — this will be proved in Theorem 5.6(a), cf. also Remark 5.5. Note, however, that $e_k \in \mathbb{C}\mathcal{G}$ is trivially satisfied for the restricted Tutte polynomial $Z_G(q, v_0)$ and hence also for the chromatic polynomial.

In Theorem 4.6, we could have required that the power sums p_k of the roots rather than the coefficients e_k be in $\mathbb{C}\mathcal{G}$. These conditions are equivalent by the Newton–Girard–Waring formulas. In most cases, it is easier to check the condition for the coefficients, but not always: consider the characteristic polynomial of the adjacency matrix of the graph G.

Proof of Theorem 4.6. This is a suitably rephrased version of the corresponding proof of Abért and Hubai.

(a) We know from elementary algebra that each p_k can be expressed as a polynomial in the e_i, and thus in the functions $H(\cdot)$. By Corollary 3.4, $\mathbb{C}\mathcal{G}$ is a ring, so this polynomial can be rewritten as a finite linear combination of the functions $H(\cdot)$. From Theorem 4.3 we know that this finite linear combination consists of terms where the graphs H are connected. Hence $p_k \in \mathbb{C}\mathcal{C}$.

Let $g(z)$ be continuous on K and harmonic on the interior. We need to prove that

$$\int_K g(z) d\mu_{G_n}(z)$$

is convergent. Choose any $\varepsilon > 0$. There exists a polynomial

$$h(z) = \sum_{k=0}^{M} a_k z^k$$

such that

$$|g(z) - \Re h(z)| \leq \varepsilon$$

for all $z \in K$, see for example [4], Lemma 3. Thus,

$$\left| \int_K g(z) d\mu_G(z) - \int_K \Re h(z) d\mu_G(z) \right| \leq \varepsilon$$

for all graphs G. Now we have

$$\int_K h(z) d\mu_G(z) = \sum_{k=0}^{M} a_k \int_K z^k d\mu_G(z) = \sum_{k=0}^{M} a_k \frac{p_k(G)}{|V(G)|}.$$

Since (G_n) was Benjamini-Schramm convergent we have that

$$\frac{p_k(G_n)}{|V(G_n)|}$$

is convergent for any fixed k, and therefore so is

$$\int_K h(z) d\mu_{G_n}(z).$$

Hence

$$(\limsup - \liminf) \int_K g(z) d\mu_{G_n}(z) \leq 2\varepsilon.$$

This holds for all $\varepsilon > 0$, so the integral converges. This completes the proof of the first statement.

For the parametric version, fix ε and note that any $\xi \in \Omega$ has a neighborhood for the points of which a common polynomial $h(z)$ can be used in the above argument. Therefore, the convergence of the integral (as $n \to \infty$) is locally uniform and hence the limit function is harmonic in ξ.

(b) We need to prove locally uniform convergence of $t_n(\xi)$. Put $\Omega = \mathbb{C} \setminus K$, and let $g(z, \xi) = \log |\xi - z|$ on $K \times \Omega$.

By (a), we have that

$$\int_K g(z, \xi) d\mu_{G_n}(z)$$

converges locally uniformly in $\xi \in \Omega$. Thus

$$t_n(\xi) = \frac{\log |f(G_n, \xi)|}{|V(G_n)|} = \frac{\sum\limits_{\lambda \text{ root}} \log |\xi - \lambda|}{|V(G_n)|} =$$

$$= \int_K \log |\xi - z| d\mu_{G_n}(z) = \int_K g(z, \xi) d\mu_{G_n}(z)$$

is locally uniformly convergent as a function of ξ. Since $t_n(\xi)$ is a harmonic function by its very definition, the harmonicity of $\lim t_n(\xi)$ follows from locally uniform convergence. $\square$

5. Graph polynomials of exponential type

In this section we study graph polynomials of exponential type. We divided this section into two subsections. In Subsection 5.1 we describe some fundamental properties of these polynomials. In Subsection 5.2 we prove the generalization of Sokal's theorem.

5.1. **Fundamental properties of graph polynomials of exponential type.**

Recall that the graph polynomial f is *of exponential type* if $f(\emptyset, x) = 1$ and for every graph $G = (V(G), E(G))$ we have

$$\sum_{S_1 \uplus S_2 = V(G)} f(S_1, x) f(S_2, y) = f(G, x + y), \tag{5.1}$$

where $f(S_1, x) = f(G[S_1], x), f(S_2, y) = f(G[S_2], y)$ are the polynomials of the subgraphs of G induced by the sets S_1 and S_2, respectively.

Note that Gus Wiseman [16] calls these graph polynomials *binomial-type.*

We will see that the chromatic polynomial is a graph polynomial of exponential type. In fact, this class contains surprisingly many known graph polynomials including the Laplacian characteristic polynomial and a modified version of the matching polynomial.

Our first aim is to describe graph polynomials of exponential type. It will turn out that if we have a function b from the class of graphs with at least one vertex to the complex numbers then this function determines a graph polynomial of exponential type and vice-versa every graph polynomial of

exponential type determines such a function. The next theorem makes this statement more precise.

Theorem 5.1. *Let b be a complex-valued function on the class of graphs on non-empty vertex sets. Let us define the graph polynomial f_b as follows. Set*

$$a_k(G) = \sum_{\{S_1,S_2,\dots,S_k\}\in\mathcal{P}_k} b(S_1)b(S_2)\dots b(S_k),$$

where the summation is over the set $\mathcal{P}_k$ of all partitions of $V(G)$ into exactly k non-empty sets. Then let

$$f_b(G,x) = \sum_{k=1}^{n} a_k(G)x^k,$$

where $n = |V(G)|$. Then

(a) *For any function b, the graph polynomial $f_b(G,x)$ is of exponential type.*
(b) *For any graph polynomial f of exponential type, there exists a graph function b such that $f(G,x) = f_b(G,x)$. More precisely, if $b(G) = a_1(G)$ is the coefficient of x^1 in $f(G,x)$, then $f = f_b$.*

Proof. (a) This is a formal computation:

$$\sum_{S_1\uplus S_2=V(G)} f_b(S_1,x)f_b(S_2,y) = \sum_{S_1\uplus S_2=V(G)} \left(\sum_{k=1}^{n} a_k(S_1)x^k\right)\left(\sum_{\ell=1}^{n} a_\ell(S_2)y^\ell\right) =$$

$$= \sum_{S_1\uplus S_2=V(G)} \left(\sum_{k=1}^{n}\left(\sum_{\{R_1,R_2,\dots,R_k\}\in\mathcal{P}_k(S_1)} b(R_1)b(R_2)\dots b(R_k)\right)x^k\right)\cdot$$

$$\cdot\left(\sum_{\ell=1}^{n}\left(\sum_{\{T_1,T_2,\dots,T_\ell\}\in\mathcal{P}_\ell(S_2)} b(T_1)b(T_2)\dots b(T_\ell)\right)y^\ell\right) =$$

$$= \sum_{r=1}^{n}\left(\sum_{\{Q_1,Q_2,\dots,Q_r\}\in\mathcal{P}_r(G)} b(Q_1)\dots b(Q_r)\right)\left(\sum_{i=1}^{r}\binom{r}{i}x^i y^{r-i}\right) =$$

$$= \sum_{r=1}^{n} a_r(G)(x+y)^r = f_b(G,x+y).$$

Hence we have proved part (a).

(b) The following proof is due to Gábor Tardos [15]. We prove the statement by induction on the number of vertices. The claim is trivial for the empty graph. Observe that by induction

$$f(G,x+y) - f(G,x) - f(G,y) = \sum_{\substack{S_1\uplus S_2=V(G)\\ S_1,S_2\neq\emptyset}} f(S_1,x)f(S_2,y) =$$

$$= \sum_{\substack{S_1\uplus S_2=V(G)\\ S_1,S_2\neq\emptyset}} f_b(S_1,x)f_b(S_2,y) = f_b(G,x+y) - f_b(G,x) - f_b(G,y).$$

In the last step we used the fact that f_b is exponential-type by part (a). Hence for the polynomial $g(x) = f(G,x) - f_b(G,x)$ we have

$$g(x+y) = g(x) + g(y).$$

Thus $g(x)$ is linear: $g(x) = cx$. On the other hand, $b(G) = a_1(G)$ is defined as the coefficient of x^1 in $f(G,x)$, whence $c = 0$. This completes the proof. $\square$

Remark 5.2. In the "nice cases" we have $f(K_1, x) = x$ or in other words, $b(K_1) = 1$ implying that f is monic, but this is not necessarily true in general.

Theorem 5.3. *Let $f_b(G,x)$ be a graph polynomial of exponential type. Then f_b is multiplicative if and only if b vanishes on non-connected graphs.*

Proof. Since the constant term of an exponential type polynomial is 0 for every graph with at least one vertex, the condition is necessary: if $H = H_1 \uplus H_2$ then $f_b(H) = f_b(H_1)f_b(H_2)$ implies that $b(H) = 0$.

On the other hand, if $b(H) = 0$ for all non-connected graphs then from Theorem 5.1 we see that

$$a_k(H_1 \uplus H_2) = \sum_{j=1}^{k} a_j(H_1)a_{k-j}(H_2)$$

which means that $f(H_1 \uplus H_2, x) = f(H_1,x)f(H_2,x)$. $\square$

Remark 5.4. Surprisingly, the class of multiplicative graph polynomials of bounded exponential type contains other natural graph polynomials besides the chromatic polynomial. We give some examples of these polynomials with their function b without proof.

- Let $M(G,x) = x^n - m_1(G)x^{n-1} + m_2(G)x^{n-2} - m_3(G)x^{n-3} + \dots$ be the (modified) matching polynomial [6, 7, 8] where $m_k(G)$ is the number of matchings of size k. Then $M(G,x)$ is of exponential type, where the function b_m satisfies $b_m(K_1) = 1$, $b_m(K_2) = -1$ and $b_m(H) = 0$ otherwise.

- Let $h(G,x) = \sum_{k=1}^{n}(-1)^{n-k}a_k(G)x^k$ be the adjoint polynomial, where $a_k(G)$ is the number of ways one can cover the vertex set of G by k vertex disjoint complete graphs. Then $h(G,x)$ is of exponential type, where the function b_h satisfies $b_h(K_n) = (-1)^{n-1}$ for the complete graphs K_n and $b_h(H) = 0$ otherwise.

- Let $L(G,x)$ be the characteristic polynomial of the Laplacian matrix, for the sake of brevity we will call it the Laplacian polynomial. The Laplacian matrix $L(G)$ is defined as follows: $L(G)_{ii} = d(i)$ is the degree of the vertex i and if $i \neq j$ then $L(G)_{ij}$ is (-1) times the number of edges connecting the vertices i and j. Then $L(G,x)$ is of exponential type, where the function b_L satisfies $b_L(G) = (-1)^{|V(G)|-1}|V(G)|\tau(G)$, where $\tau(G)$ is the number of spanning trees of the graph G.

- The polynomial $Z_G(q, v_0)$ is of exponential type for every fixed v_0. In fact, one can easily prove the following identity for the multivariate version of the

Tutte polynomial [13]:

$$\sum_{S\subseteq V(G)} Z_{G[S]}(q_1,\underline{v})Z_{G[V\setminus S]}(q_2,\underline{v}) = Z_G(q_1+q_2,\underline{v}).$$

Remark 5.5. To prove Theorem 1.10, we only need to be able to apply Theorem 4.6. For that, we need to prove that for f isomorphism-invariant and of exponential type, $e_k \in \mathbb{C}\mathcal{G}$ holds. However, we include the following more precise statements for the sake of completeness. Recall that if $\mathcal{H}$ is a class of graphs then $\mathbb{C}\mathcal{H}$ is the vector space generated by the functions $H(\cdot)$ ($H \in \mathcal{H}$). The following theorem asserts that for exponential-type graph polynomials, the coefficients e_k and power sums p_k are in $\mathbb{C}\mathcal{H}_k$, where $\mathcal{H}_k$ is a very special class of graphs.

Theorem 5.6. *Let*

$$f(G,x) = \sum_{k=0}^{n}(-1)^k e_k(G)x^{n-k}$$

be an isomorphism-invariant monic graph polynomial of exponential type, where $n = |V(G)|$*. Let* $k \geq 1$*. Define* $p_k(G)$ *to be the* k*-th power sum of the roots of* $f(G,x)$*.*

(a) *We have*

$$e_k \in \mathbb{C}\{H : k+1 \leq |V(H)| \leq 2k, |V(H)| - k(H) \leq k\} \tag{5.2}$$

and

$$p_k \in \mathbb{C}\{H : 2 \leq |V(H)| \leq 2k, |V(H)| - k(H) \leq k\}. \tag{5.3}$$

(b) *If, in addition,* f *is multiplicative, then*

$$e_k \in \mathbb{C}\{H : H \textit{ has no isolated points and } |V(H)| - k(H) = k\} \tag{5.4}$$

and

$$p_k \in \mathbb{C}\{H : H \textit{ is connected and } 2 \leq |V(H)| \leq k+1\}. \tag{5.5}$$

Proof. We will apply Theorem 5.1. Note that

$$e_k(G) = (-1)^k a_{n-k}(G).$$

By Theorem 5.1 we have

$$a_{n-k}(G) = \sum_{\{S_1,S_2,\dots,S_{n-k}\}\in\mathcal{P}_{n-k}} b(S_1)\cdots b(S_{n-k}),$$

where the summation runs over the set $\mathcal{P}_{n-k}$ of all partitions of $V(G)$ into exactly $n-k$ non-empty sets. Let us consider the partition $\pi = \{S_1,\dots,S_{n-k}\}$. Let it have $n-l$ singletons and $l-k$ bigger parts. Then the union R of these bigger parts has l points and $k+1 \leq l \leq 2k$. We group the terms of the last sum according to the isomorphism class of $G[R]$:

$$a_{n-k}(G) = \sum_{k+1\leq|V(H)|\leq 2k} H^*(G) \sum_{\bar{\pi}=\{S_1,S_2,\dots,S_{|H|-k}\}} b(S_1)\cdots b(S_{|H|-k}), \tag{5.6}$$

where $\bar{\pi}$ runs over the partitions of $V(H)$ into $|H|-k$ sets of at least 2 elements. By Fact 3.2 this can be rewritten as

$$a_{n-k}(G)=\sum_{k+1\le|V(H)|\le 2k} c_k(H)\cdot H(G), \tag{5.7}$$

where $c_k(H)\in\mathbb{C}$. This already proves that $e_k\in\mathbb{C}\mathcal{G}$. Note that formula (5.7) holds for $k=0$ as well, except that we no longer have $k+1\le|V(H)|$. More precisely, $c_0(\emptyset)=1$ and $c_0(H)=0$ for all non-empty graphs H.

We need to prove that $c_k(H)=0$ unless $|V(H)|-k(H)\le k$. The defining identity (5.1) of exponential type and formula (5.7) yields

$$\sum_k\sum_H c_k(H)\cdot H(G)(x+y)^{n-k}=$$

$$=\sum_{S\subseteq V(G)}\sum_i\sum_{H'} c_i(H')\cdot H'(S)x^{|S|-i}\sum_j\sum_{H''} c_j(H'')\cdot H''(G-S)y^{n-|S|-j}=$$

$$=\sum_{H'}\sum_{H''}\sum_i\sum_j c_i(H')c_j(H'')\cdot(H'\uplus H'')(G)(x+y)^{n-|H'|-|H''|}x^{|H'|-i}y^{|H''|-j}.$$

We consider the part that is homogeneous of degree $n-k$ in x and y:

$$\sum_H c_k(H)\cdot H(G)(x+y)^{n-k}=$$

$$=\sum_{H'}\sum_{H''}\sum_{i+j=k} c_i(H')c_j(H'')\cdot(H'\uplus H'')(G)(x+y)^{n-|H'|-|H''|}x^{|H'|-i}y^{|H''|-j}.$$

This is true for all G, therefore

$$c_k(H)(x+y)^{n-k}=\sum_{H'\uplus H''=H}\sum_{i+j=k} c_i(H')c_j(H'')(x+y)^{n-|H|}x^{|H'|-i}y^{|H''|-j}.$$

We divide by $(x+y)^{n-|H|}$ to get

$$c_k(H)(x+y)^{|H|-k}=\sum_{H'\uplus H''=H}\sum_{i+j=k} c_i(H')c_j(H'')x^{|H'|-i}y^{|H''|-j}=$$

$$=c_k(H)(x^{|H|-k}+y^{|H|-k})+\sum_{H'\uplus H''=H}\sum_{i=1}^{k-1} c_i(H')c_{k-i}(H'')x^{|H'|-i}y^{|H''|-k+i}.$$

If $|V(H)|=k+1$, then $|V(H)|-k(H)\le k$ trivially holds and there is nothing to prove. If $|V(H)|\ge k+2$ and $c_k(H)\ne 0$, then

$$0\ne c_k(H)\left((x+y)^{|H|-k}-x^{|H|-k}-y^{|H|-k}\right)=$$

$$=\sum_{H'\uplus H''=H}\sum_{i=1}^{k-1} c_i(H')c_{k-i}(H'')x^{|H'|-i}y^{|H''|-k+i}.$$

Thus there exists $1\le i\le k-1$ and a decomposition $H'\uplus H''=H$ with $c_i(H')c_{k-i}(H'')\ne 0$. By induction on k, we may assume that $|V(H')|-k(H')\le i$ and $|V(H'')|-k(H'')\le k-i$, whence $|V(H)|-k(H)\le k$. This proves the statement (5.2) of part (a).

For the statement (5.3), recall from elementary algebra that p_k is an integral linear combination of terms $\prod e_{i_j}$, where $i_j \geq 1$ and $\sum i_j = k$. The statement (5.3) thus follows from statement (5.2) and Fact 3.3.

In part (b), we assume that f is multiplicative. By Theorem 5.3, b vanishes on non-connected graphs. Thus, from equation (5.6) and Fact 3.2 we see that

$$e_k \in \mathbb{C}\{H : H \textit{ has no isolated points and } |V(H)| - k(H) \geq k\}.$$

Using (5.2), the statement (5.4) follows. The statement (5.5) is immediate from Lemma 4.3 and the statement (5.3). ☐

5.2. **Proof of Theorem 1.6 (the Sokal bound).** To prove Theorem 1.6, we will need the concept of the multivariate independence polynomial.

Definition 5.7. Let $w : V(G) \to \mathbb{C}$ and let

$$I(G, \underline{w}) = \sum_{S \in \mathcal{I}} \prod_{u \in S} w_u$$

be the *multivariate independence polynomial.*

Our strategy will be the following. We express any exponential-type graph polynomial as an independence polynomial of a bigger graph. Then we use Dobrushin's lemma and its corollaries to deduce Theorem 1.6.

Theorem 5.8 (Dobrushin's Lemma)**.** *Assume that the functions $r : V(G) \to (0, 1)$ and $w : V(G) \to \mathbb{C}$ satisfy the inequalities*

$$|w_v| \leq (1 - r(v)) \prod_{(v,v') \in E(G)} r(v')$$

for every $v \in V(G)$. Then

(a) *$I(A, \underline{w}) \neq 0$ for every $A \subseteq V(G)$.*

(b)
$$\left| \frac{I(B, \underline{w})}{I(A, \underline{w})} \right| \leq \left(\prod_{u \in A \setminus B} r(u) \right)^{-1}$$
for every $B \subseteq A \subseteq V(G)$.

Proof. (The following proof is a very slight modification of Borgs' proof [2].) We prove the two statements together by induction on the size of $|A|$. Both statements are trivial if $|A| = 0$. Now assume that we have already proved both statements for sets of size smaller than $|A|$. Now let us prove them for the set A. Let $v \in A$. Recall that $N[v] = N(v) \cup \{v\}$.

$$\begin{aligned} I(A, \underline{w}) &= I(A - v, \underline{w}) + w_v I(A - N[v], \underline{w}) = \\ &= I(A - v, \underline{w}) \left(1 + w_v \frac{I(A - N[v], \underline{w})}{I(A - v, \underline{w})} \right). \end{aligned}$$

By induction $I(A - v, \underline{w}) \neq 0$ so we can indeed divide by it. By part (b) of the induction we have

$$\left| w_v \frac{I(A - N[v], \underline{w})}{I(A - v, \underline{w})} \right| \leq |w_v| \left(\prod_{v' \in (A - v) \setminus (A - N[v])} r(v') \right)^{-1} \leq$$

Now let us use the condition on w_u's. (Clearly, $(A-v)\setminus(A-N[v]) = \{v' \mid (v,v')\in E(A)\}$.)

$$\leq (1-r(v))\prod_{(v,v')\in E(G)} r(v')\left(\prod_{v'\in(A-v)\setminus(A-N[v])} r(v')\right)^{-1} = 1-r(v).$$

Hence

$$|I(A,\underline{w})| = |I(A-v,\underline{w})|\left|1+w_v\frac{I(A-N[v],\underline{w})}{I(A-v,\underline{w})}\right| \geq$$

$$\geq |I(A-v,\underline{w})|\left(1-\left|w_u\frac{I(A-N[v],\underline{w})}{I(A-v,\underline{w})}\right|\right) \geq$$

$$\geq |I(A-v,\underline{w})|r(v) > 0$$

Hence $I(A,\underline{w})\neq 0$. Finally, if $B\subset A$ then for any $v\in A\setminus B$ we have $B\subseteq A-v$ so we obtain

$$\left|\frac{I(B,\underline{w})}{I(A,\underline{w})}\right| = \left|\frac{I(B,\underline{w})}{I(A-v,\underline{w})}\right|\left|\frac{I(A-v,\underline{w})}{I(A,\underline{w})}\right| \leq$$

$$\leq \left(\prod_{u\in(A-v)\setminus B} r(u)\right)^{-1} r(v)^{-1} = \left(\prod_{u\in A\setminus B} r(u)\right)^{-1}.$$

Hence we have proved part (b) as well. □

Remark 5.9. The condition of Dobrushin's lemma resembles that of the Lovász local lemma. This is not a coincidence. Scott and Sokal [12] gave an exact form of the Lovász local lemma in terms of the independence polynomial. Although their theorem is precise, it is hard to use it in its original form. Dobrushin's lemma provides a way to obtain a useful relaxation of their theorem. In this way we recover the original Lovász local lemma.

In our application, it will be more convenient to check the condition of the following version of Dobrushin's lemma.

Corollary 5.10. *If the function $a: V(G)\to\mathbb{R}^+$ satisfies the inequality*

$$\sum_{u\in N[v]}\log\left(1+|w_u|e^{a(u)}\right)\leq a(v)$$

for each vertex $v\in V(G)$, then

$$I(G,\underline{w})\neq 0.$$

In particular, if the function $a: V(G)\to\mathbb{R}^+$ satisfies the inequality

$$\sum_{u\in N[v]}|w_u|e^{a(u)}\leq a(v)$$

for each vertex $v\in V(G)$, then

$$I(G,\underline{w})\neq 0.$$

Proof. Set $r(v) = \left(1 + |w_v|e^{a(v)}\right)^{-1}$. Then the condition

$$|w_v| \leq (1 - r(v)) \prod_{(v,v')\in E(G)} r(v')$$

can be rewritten using

$$(1 - r(v)) \prod_{(v,v')\in E(G)} r(v') = \left(1 - \frac{1}{1 + |w_v|e^{a(v)}}\right) \prod_{u\in N(v)} \frac{1}{1 + |w_u|e^{a(u)}} =$$

$$= |w_v|e^{a(v)} \prod_{u\in N[v]} \frac{1}{1 + |w_u|e^{a(u)}}.$$

Hence the condition of Dobrushin's lemma is equivalent to

$$\sum_{u\in N[v]} \log\left(1 + |w_u|e^{a(u)}\right) \leq a(v).$$

The second statement simply follows from the inequality $\log(1 + s) \leq s$ for $s \geq 0$. □

We can express a (monic) graph polynomial of exponential type as a multivariate independence polynomial as follows.

Let us define the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ as follows. The vertex set $\tilde{V}$ consists of the subsets of $V(G)$ of size at least 2. If $S_1, S_2 \subseteq V(G)$ such that $|S_1|, |S_2| \geq 2$ then let $(S_1, S_2) \in \tilde{E}$ if and only if $S_1 \cap S_2 \neq \emptyset$. Hence the independent sets of $\tilde{G}$ are those subsets $S_1, \dots, S_k \subset V(G)$ for which $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq k$ and $|S_i| \geq 2$. In what follows we will call $\{S_1, S_2, \dots, S_k\}$ a subpartition. The set of subpartitions will be denoted by $\mathcal{P}'(G)$.

Let us define the function (or weights) of the vertices of $\tilde{G}$ as follows. Set

$$w(S) = \frac{b(S)}{q^{|S|-1}}.$$

Note that if we have a subpartition $\{S_1, S_2, \dots, S_k\}$ then by dividing $V(G) \setminus \bigcup_{i=1}^k S_i$ into 1-element sets we get a partition of $V(G)$ into

$$k + n - \sum_{i=1}^{k} |S_i| = n - \sum_{i=1}^{k} (|S_i| - 1)$$

sets. Then

$$I(\tilde{G}, \underline{w}) = \sum_{\{S_1,\dots,S_k\}\in\mathcal{P}'(G)} \prod_{i=1}^{k} \frac{b(S_i)}{q^{|S_i|-1}} = q^{-n} f_b(G, q).$$

Hence if we find a zero-free region for $I(\tilde{G}, \underline{w})$ then we find a zero-free region for $f_b(G, x)$.

Now we will specialize Corollary 5.10 to our case.

Lemma 5.11. *Assume that for some positive real number a we have*

$$\sum_{\substack{S\in\tilde{V}\\ v\in S}} |b(S)| \left(\frac{e^a}{|q|}\right)^{|S|-1} \leq ae^{-a}$$

for every $v \in V(G)$. Then $f_b(G,q) \neq 0$.

Proof. Let us check the second condition of Corollary 5.10 for the weight function $w(S) = \frac{b(S)}{q^{|S|-1}}$ and the function $a(S) = a|S|$. We need that

$$\sum_{S:(T,S)\in \tilde{E}} |w(S)|e^{a|S|} \leq a|T|$$

for every $T \in \tilde{V}$. (Recall that $(T,S) \in \tilde{E}$ means that $S \cap T \neq \emptyset$.) This is indeed true since

$$\sum_{S:(T,S)\in \tilde{E}} |w(S)|e^{a|S|} \leq \sum_{v\in T}\sum_{v\in S} |w(S)|e^{a|S|} =$$

$$= \sum_{v\in T}\sum_{v\in S} \frac{|b(S)|}{q^{|S|-1}}e^{a|S|} = \sum_{v\in T} e^a \sum_{v\in S} |b(S)| \left(\frac{e^a}{|q|}\right)^{|S|-1} \leq$$

By the assumption of the lemma, the inner sum can be bounded above by ae^{-a}. The whole expression is therefore

$$\leq \sum_{v\in T} e^a(ae^{-a}) = a|T|,$$

as claimed. □

Now we are ready to prove Theorem 1.6.

Proof. Since f is of bounded exponential type, we have

$$\sum_{\substack{S:v\in S\\ |S|=t}} |b(S)| \leq R(\Delta)^{t-1},$$

whence

$$\sum_{\substack{S\in\tilde{V}\\ v\in S}} |b(S)| \left(\frac{e^a}{|q|}\right)^{|S|-1} = \sum_{t=2}^{\infty} \left(\sum_{\substack{S:v\in S\\ |S|=t}} |b(S)|\right) \left(\frac{e^a}{|q|}\right)^{t-1} \leq$$

$$\leq \sum_{t=2}^{\infty} R(\Delta)^{t-1} \left(\frac{e^a}{|q|}\right)^{t-1} = \frac{1}{1-\frac{R(\Delta)e^a}{|q|}} - 1$$

So the polynomial $f(G,x)$ does not vanish at q if

$$\frac{1}{1-\frac{R(\Delta)e^a}{|q|}} \leq 1 + ae^{-a}.$$

This is satisfied if

$$|q| \geq (e^a + e^{2a}/a)R(\Delta).$$

Hence the roots of the polynomial $f(G,z)$ have absolute value $< c \cdot R(\Delta)$, where

$$c = \min_{a\geq 0}(e^a + e^{2a}/a) \approx 7.0319,$$

and the minimalizing $a \approx 0.4381$. □

Remark 5.12. The condition of Definition 1.4 seems to be frightening, but in several cases it is easy to check. For the matching polynomial or the adjoint polynomial $R(\Delta) = \Delta$ trivially satisfies the inequality. We get a slightly better bound if we apply Lemma 5.11 directly. On the other hand, it can be proved that the sharp upper bound for the roots is $4(\Delta - 1)$ in both cases. For the matching polynomial this is proved in [8], for the adjoint polynomial this is proved in [5].

6. 2-multiplicativity and 2-connected graphs

In a strict sense this part is not connected to the main objective of this paper, but it makes the picture more complete for the most interesting graph polynomials.

We know that the chromatic polynomial and the restricted Tutte polynomial $Z(q, v_0)$ for any fixed v_0 are multiplicative. In fact, they are even 2-multiplicative in the following sense.

Definition 6.1. We say that the graph polynomial f is *2-multiplicative* if it is multiplicative and

$$xf(G_1 \cup G_2, x) = f(G_1, x)f(G_2, x),$$

where $G_1 \cup G_2$ is any union of the graphs G_1 and G_2 such that they have exactly one vertex in common.

We have seen that the multiplicativity of a graph polynomial implies that the coefficients of non-connected graphs vanish in the k-th power sum p_k: see Lemma 4.3. The following statements show that 2-multiplicativity implies that the coefficients of non-2-connected graphs vanish in the k-th power sum p_k. Recall that $\mathcal{C}_2$ denotes the class of 2-connected graphs.

Lemma 6.2. *(2-multiplicative lemma.) Assume that f is a 2-multiplicative graph polynomial such that $f(G, x)$ is not the zero polynomial for any graph G. For some $k \geq 1$, assume that the k-th power sum of the roots $p_k \in \mathbb{C}\mathcal{G}$. Then $p_k \in \mathbb{C}\mathcal{C}_2$.*

Naturally, the proof of Lemma 6.2 relies on the corresponding analogue of Lemma 4.2.

Definition 6.3. A function p defined on graphs is *2-additive* if

$$p(G_1 \cup G_2) = p(G_1) + p(G_2)$$

holds for the union of any two graphs G_1 and G_2 that have at most one vertex in common.

Lemma 6.4. *(2-additive lemma.) A function $p \in A\mathcal{G}$ is 2-additive if and only if $p \in A\mathcal{C}_2$.*

The proof of this lemma is the straightforward analogue of the proof of Lemma 4.2.

Proof of Lemma 6.2. The function $G \mapsto p_k(G)$ is 2-additive, so the statement follows from Lemma 6.4. $\square$

For the sake of completeness, we include the analogues of those theorems where multiplicativity played some role. The following theorem gives a description of 2-multiplicative graph polynomials of exponential type.

Theorem 6.5. *Let $f_b(G,x)$ be a graph polynomial of exponential type. Then f_b is 2-multiplicative if and only if*

(i) *b vanishes on non-connected graphs and*
(ii) *$b(H_1 \cup H_2) = b(H_1)b(H_2)$ whenever H_1 and H_2 have exactly one vertex in common.*

Proof. Since 2-multiplicativity implies multiplicativity, the condition (i) is necessary. For (ii), note that if $H = H_1 \cup H_2$ and $|V(H_1) \cap V(H_2)| = 1$ then by taking the coefficient of x^2 in $xf_b(H) = f_b(H_1)f_b(H_2)$ we get the statement.

On the other hand, if (i) and (ii) hold, then f_b is multiplicative and from Theorem 5.1 we see that if $|V(H_1) \cap V(H_2)| = 1$ then

$$a_k(H_1 \cup H_2) = \sum_{j=1}^{k} a_j(H_1)a_{k+1-j}(H_2)$$

which means that $xf(H_1 \cup H_2, x) = f(H_1, x)f(H_2, x)$. □

The following theorem is the natural "third part" of Theorem 5.6.

Theorem 6.6. *Let*

$$f(G,x) = \sum_{k=0}^{n} (-1)^k e_k(G) x^{n-k}$$

be an isomorphism-invariant monic graph polynomial of exponential type, where $n = |V(G)|$. Let $k \geq 1$. Define $p_k(G)$ to be the k-th power sum of the roots of $f(G,x)$. If f is 2-multiplicative, then

$$p_k \in \mathbb{C}\{H : H \text{ is 2-connected and } 2 \leq |V(H)| \leq k+1\}.$$

Proof. The statement follows from Lemma 6.2 and (5.5). □

Remark 6.7. It is also interesting to note that if we consider the chromatic polynomial, then in the expression

$$p_k(\cdot) = \sum_H c_k(H)H(\cdot),$$

all coefficients $c_k(H)$ are integers. In fact, this is always true if the parameters $b(\cdot)$ defining the exponential type graph polynomial are integers. So it is also true for the matching, adjoint, Laplacian polynomials. This can be seen from the proof of Theorem 5.6.

Moreover, by looking at that proof more closely, it is also easy to obtain an explicit formula for $c_k(H)$. Let f be an isomorphism-invariant, multiplicative monic graph polynomial of exponential type. By Theorem 5.6(b), we have

$$(-1)^i e_i(G) = \sum_H c(H)H(G),$$

where H runs over graphs without isolates satisfying $|V(H)| - k(H) = i$. Then, for $k \geq 1$,

$$c_k(H) = k \sum_{H_1,\ldots,H_m} (-1)^m \frac{(m-1)!}{\prod_i m_i!} \prod_{j=1}^{m} c(H_j). \tag{6.1}$$

The summation is over sequences $H_1, \ldots, H_m$ of isolate-free subgraphs of H which satisfy $V(H_1)\cup\cdots\cup V(H_m) = V(H)$, $E(H_1)\cup\cdots\cup E(H_m) = E(H)$ and $\sum im_i = k$, where m_i is the number of indices j such that $|V(H_j)| - k(H_j) = i$.

By Theorem 6.6, the sum in (6.1) is zero unless H is 2–connected and $2 \leq |V(H)| \leq k+1$.

When $f(G,q)$ is the Tutte polynomial $Z_G(q,v)$, we have $c(H) = v^{|E(H)|}$, so

$$c_k(H) = k \sum_{H_1,\ldots,H_m} (-1)^m v^{\sum |E(H_j)|} \frac{(m-1)!}{\prod m_i!}. \tag{6.2}$$

For the chromatic polynomial, we substitute $v = -1$ in (6.2).

7. Concluding remarks

Clearly, the conditions of Theorem 4.6 are very weak and most of the well-known graph polynomials satisfy them. The class of multiplicative graph polynomials of bounded exponential type already contains the matching polynomial, the chromatic polynomial (in fact, all restricted Tutte polynomials) and the Laplacian characteristic polynomial. On the other hand, the characteristic polynomial of the adjacency matrix is not of exponential type, but trivially satisfies all conditions: it is multiplicative, all its roots have absolute value at most Δ and $p_k(G) = \hom(C_k, G)$ (the number of closed walks of length k).

It is not known how the condition $\xi \notin K$ in Theorem 4.6(b) can be relaxed. R. Lyons [10] proved that if (G_n) is Benjamini-Schramm convergent, then $(\log \tau(G_n))/|V(G_n)|$ is convergent, where $\tau(G_n)$ denotes the number of spanning trees of the graph G_n. This result can be expressed as a result concerning a special value of the Tutte polynomial, namely $\tau(G) = T(G,1,1)$. It is clearly not a special case of our theorems, because this point is deep inside the disc K given by the Sokal bound.

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